CERN-PH-TH/2004-198

Rotational inhomogeneities from pre-big bang?

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Abstract

The evolution of the rotational inhomogeneities is investigated in the specific framework of four-dimensional pre-big bang models. While minimal (dilaton-driven) scenarios do not lead to rotational fluctuations, in the case of non-minimal (string-driven) models, fluid sources are present in the pre-big bang phase. The rotational modes of the geometry, coupled to the divergenceless part of the velocity field, can then be amplified depending upon the value of the barotropic index of the perfect fluids. In the light of a possible production of rotational inhomogeneities, solutions describing the coupled evolution of the dilaton field and of the fluid sources are scrutinized in both the string and Einstein frames. In semi-realistic scenarios, where the curvature divergences are regularized by means of a non-local dilaton potential, the rotational inhomogeneities are amplified during the pre-big bang phase but they decay later on. Similar analyses can also be performed when a contraction occurs directly in the string frame metric.

1 Introduction

If a four-dimensional Friedmann–Robertson–Walker (FRW) Universe is expanding, the rotational fluctuations of the geometry are not amplified. This statement is well known since the early analyses of Lifshitz and Khalatnikov (see section 9 of Ref. [1]).

The results of Ref. [1] assume four important premises:

- the Universe is *four-dimensional* and it is described by a FRW line element;
- the matter sources driving the evolution of the geometry are *perfect relativistic fluids* with barotropic equation of state;
- the Universe is always *expanding*;
- the connection between the evolution of the matter sources and the evolution of the geometry is provided by *general relativity.*

If these four premises are rigorously verified the conclusions of Ref. [1] follow by studying the evolution of the rotational inhomogeneities of the metric in a FRW background. The rotational inhomogeneities of the metric are parametrized in terms of two divergenceless vectors in three spatial dimensions, leading, overall, to four independent degrees of freedom (see section 2 below for a more formal discussion). The rotational fluctuations of the geometry are also coupled, through Einstein equations, to the divergenceless part of the velocity field supported by the presence of a perfect barotropic fluid.

If some of the mentioned premises are not verified, there is the logical possibility that the rotational inhomogeneities are amplified. For instance Grishchuk [2] analysed the situation where the Universe is not filled by a perfect barotropic fluid. Indeed, in the model of Ref. [2], the early stages of the life of the Universe are described by a medium characterized by a non-vanishing torque force. The resulting rotational modes are, in this case, copiously produced.

Looking at the hypotheses listed above, there could also be, in principle, other ways of obtaining rotational modes of the geometry and of the velocity field in reasonable cosmological scenarios. For instance, it could happen that the geometry is not purely four-dimensional or that the dynamical equations connecting the evolution of the matter sources to that of the geometry do not follow from general relativity but from some other non-Einsteinian theory of gravity.

Indeed, if the geometry has more than four dimensions, the rotational degrees of freedom are more numerous than in the four-dimensional case. Furthermore, depending on the

specific theory, the evolution equations of the rotational degrees of freedom are qualitatively different with respect to the four-dimensional case. This possible perspective was recently invoked in [3], where the evolution of the vector modes of a five-dimensional theory have been analysed in the context of a regular model of dynamical dimensional reduction.

Yet another way of amplifying dynamically rotational inhomogeneities would be to assume that the Universe was not always expanding but that there have been long periods of contraction. Different (unconventional) paradigms of the early Universe may realize, in one way or another, this requirement. For instance, various incarnations of string cosmological scenarios, such as the pre-big bang [4, 5, 6] and the ekpyrotic [7, 8, 9] paradigms, assume that in the past history of the Universe a rather long contracting phase took place. To be more specific, the pre-big bang dynamics is described by a phase of accelerated expansion in the string[1] frame but this evolution becomes, indeed, an accelerated contraction in the more conventional Einstein frame [10, 11]. On the contrary, in some explicit realization of the ekpyrotic scenarios, a slow contraction takes place already in the string frame description.

Concerning the issue raised in the previous paragraph, a relevant technical remark is in order. The possibility of defining different frames is inherent to the non-Einsteinian nature of the low-energy string effective action. It should be borne in mind that the first three of the four assumptions mentioned at the beginning of this investigation are not independent in the sense that the word *contraction* has a definite meaning only if the action of the underlying theory is specified. Thus, provided the theory is not of Einstein-Hilbert type, there is no surprise if the rotational modes are amplified in an expanding Universe.

There is an important physical distinction to be made when discussing the amplification of fluctuations in a cosmological context. The amplification can be either *adiabatic* or *super-adiabatic*. The adiabatic amplification implies the growth of the amplitude. The super-adiabatic amplification implies that the evolution equations exibit an instability for a given range of Fourier modes. If the vector modes are the ones of a four-dimensional Universe, the equations of motion, as it will be shown, only allow the possibility of adiabatic amplification (unless specific couplings to som medium with non-vanishing torque force are envisaged as in Ref. [2]). If, however, we go to higher dimensions (for instance in 5 space-time dimensions),

[1] The string and Einstein frames are two different parametrizations of the action of the theory and are connected by simple (local) field redefinitions involving the metric and the dilaton field. In particular, in the string frame action (see section 2) the dilaton field is directly coupled to the Ricci scalar, while in the Einstein frame action the dilaton is not directly coupled to the Einstein–Hilbert term (see section 3). Notice that, as far as the description is concerned, both the Einstein and the string frames can be employed for actual calculations and the results will be the same. However, at a practical level, there are cases where one frame is more convenient than the other. Interesting arguments in favour of the string frame were put forward since the centre of mass of fundamental test strings follows geodesics in the string and not in the Einstein frame [12].

there, one more physical vector mode appears; its evolution exhibits clearly the phenomenon of super-adiabatic amplification [6].

Recently, Battefeld and Brandenberger [13] argued that vector modes of the geometry can be produced in contracting models. To be more specific, the argument of [13] assumes that the matter content of the Universe is *only* given by a perfect relativistic fluid and that the theory is of Einstein-Hilbert type. If the matter content of the Universe would consist only of a scalar field driving a contraction, the effective velocity field will be automatically irrotational. This is exactly the situation of minimal pre-big bang models.

The authors of Ref. [13] did not consider any specific regular solution connecting, in their framework, the contracting to the expanding phase. This remark has already been introduced in [3] where it was, almost incidentally, argued that a complete study of string-driven pre-big bang models should be performed in order to support the idea of amplification of the rotational modes of the geometry. In this spirit, one of the sections of [3] dealt with a simplified four-dimensional model whose analysis showed that vector modes of the geometry may well be amplified but decay later on.

The motivation of the present paper is to present a plausible systematic analysis of the possible production of rotational inhomogeneities in four-dimensional pre-big bang models. String cosmological models incorporating the evolution of fluid sources are often qualified string-driven since, in these cases, inflation can be obtained also by studying the evolution of a gas of fundamental strings [14].

Before plunging into the analysis it is appropriate to mention some investigations dicussing rotational fluctuations in a perspective different from the one employed in the present paper. In Ref. [15], Gödel solutions have been investigated in the low-energy string effective action including string tension corrections. In these solutions the magnitude of the vorticity vector is proportional to the inverse of the string tension. In [16, 17, 18] where the evolution of the vorticity was studied in Friedmann-Robertson-Walker models. The approach followed by these authors is to perturb around a spherically symmetric profile of the geometry. The approach reported in the present investigation is connected with the Bardeen formalism and it is then different from the one of Ref. [16, 17, 18] . The results of the two approaches seem to be in agreement since in Ref. [16, 17, 18] is also found that the rotation parameter depends upon an inverse power of the scale-factor . Finally, in [19] the possible rôle of rotation was investigated in the context of Bianchi type-IX models.

It is also appropriate to mention some early (very interesting) references dealing with stiff fluids in the early Universe. In [21] (and, partially also in [22]) one of the unusual features of fluids stiffer than radiation was pointed out: the rotational velocity increases as the Universe expands. A reprise of this theme appears to be Ref. [20] where similar

considerations are discussed in the context of the so-called holographic cosmology.

The present study is organized as follows. In section 2 various (singular) solutions of the low-energy string effective action will be investigated in the presence of perfect fluids with barotropic equation of state. The evolution of the rotational inhomogeneities will then be obtained. In section 3 the Einstein frame picture will be presented in the light of the evolution of the rotational modes. Then, in section 4, non-singular models of pre-big bang will be specifically discussed and the corresponding evolution of the rotational inhomogeneities derived. Finally, section 5 contains some concluding (critical) remarks.

2 Pre-big bang and ekpyrotic initial conditions

Consider, to begin with, the low-energy string effective action in the presence of fluid sources in four space-time dimensions:

$$S = -\frac{1}{\lambda_{\rm s}^2}\int d^4x\sqrt{-G}e^{-\varphi}[R + (\partial\varphi)^2 + W(\varphi)] + S_{\rm m}. \tag{2.1}$$

In Eq. (2.1) $\lambda_{\rm s}$ is the string length scale and $W(\varphi)$ is a (local) potential depending on the four-dimansional dilaton field φ; the term $S_{\rm m}$ represents the contribution to the total action coming from perfect barotropic fluids. The compact notation

$$(\partial\varphi)^2 = G^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi, \quad \nabla^2\varphi = G^{\alpha\beta}\nabla_\alpha\nabla_\beta\varphi = G^{\alpha\beta}(\partial_\alpha\partial_\beta\varphi - \Gamma^{\sigma}_{\alpha\beta}\partial_\sigma\varphi), \tag{2.2}$$

will be also employed throughout this investigation. The possible coupling of the dilaton field to the matter sources as well as the possible presence of an antisymmetric tensor field have been neglected. As already stressed in the introduction, the action (2.1) has been written in what is customarily defined as the string frame metric.

The functional derivation of (2.1) with respect to $G_{\mu\nu}$ and φ leads to the well-known equations:

$$\mathcal{G}_{\mu\nu} + \nabla_\mu\nabla_\nu\varphi + \frac{1}{2}G_{\mu\nu}[(\partial\varphi)^2 - 2\nabla^2\varphi - W(\varphi)] = \lambda_s^2 e^{\varphi}T_{\mu\nu}, \tag{2.3}$$

$$R + 2\nabla^2\varphi - (\partial\varphi)^2 + W - \frac{\partial W}{\partial\varphi} = 0, \tag{2.4}$$

$$\nabla_\mu T^{\mu\nu} = 0, \tag{2.5}$$

where $T_{\mu\nu}$ is the energy-momentum tensor of the fluid sources and $\mathcal{G}_{\mu\nu} = R_{\mu\nu} - 1/2G_{\mu\nu}R$ the usual Einstein tensor.

Combining Eqs. (2.3) and (2.4) the following useful equation

$$R_{\mu\nu} + \nabla_\mu\nabla_\nu\varphi - \frac{1}{2}\frac{\partial W}{\partial\varphi}G_{\mu\nu} = \lambda_{\rm s}^2 e^{\varphi}T_{\mu\nu} \tag{2.6}$$

can be obtained. In a spatially flat FRW geometry,

$$ds^2 = G_{\mu\nu} dx^\mu dx^\nu = dt^2 - a^2(t) d\vec{x}^2, \tag{2.7}$$

with homogeneous dilaton field $\varphi = \varphi(t)$, the explicit form of the background equations follows from the components of Eq. (2.3) (or (2.6)) and from Eqs. (2.4) and (2.5):

$$\dot{\varphi}^2 + 6H^2 - 6H\dot{\varphi} - W = 2\lambda_{\rm s}^2 \rho e^{\varphi}, \tag{2.8}$$

$$\dot{H} = H\dot{\varphi} - 3H^2 - \frac{\partial W}{\partial \varphi} + \lambda_{\rm s}^2 p e^{\varphi}, \tag{2.9}$$

$$2\ddot{\varphi} - 6\dot{H} - 12H^2 - \dot{\varphi}^2 + 6H\dot{\varphi} + W - \frac{\partial W}{\partial \varphi} = 0, \tag{2.10}$$

$$\dot{\rho} + 3H(\rho + p) = 0. \tag{2.11}$$

In Eqs. (2.8)–(2.11) the overdot denotes the derivation with respect to the cosmic time coordinate t, and ρ and p are the energy and pressure density of the perfect fluid whose energy-momentum tensor has the standard form

$$T_{\mu\nu} = (p + \rho) u_\mu u_\nu - p G_{\mu\nu}. \tag{2.12}$$

Since we shall assume that the fluid is barotropic, i.e. $p = \gamma\rho$, Eq. (2.11) gives, after direct integration

$$\rho = \rho_0 a^{-3(\gamma+1)}. \tag{2.13}$$

In pre-big bang models the initial conditions are customarily set for $t < 0$. Indeed we shall be looking for solutions of the system of Eqs. (2.8)–(2.11), which can then be parametrized as

$$a(t) = (-\tau)^\alpha, \tag{2.14}$$

$$\varphi(t) = \varphi_0 - \beta \ln(-\tau), \tag{2.15}$$

$$\rho = \rho_0 (-\tau)^{-3\alpha(\gamma+1)}, \tag{2.16}$$

where $\tau = t/t_0$ and where Eq. (2.16) follows by inserting Eq. (2.14) into Eq. (2.13).

Consider now the case $W(\varphi) = 0$. Inserting Eqs. (2.14)–(2.16) into Eqs. (2.8)–(2.10) the following set of relations between the parameters must be satisfied:

$$3\alpha(\gamma + 1) + \beta = 2, \tag{2.17}$$

$$\beta(2 - \beta) + 6\alpha(1 - 2\alpha - \beta) = 0, \tag{2.18}$$

and

$$\rho_0 = e^{-\varphi_0} \frac{\beta^2 + 6\alpha^2 + 6\alpha\beta}{2\lambda_{\rm s}^2} = e^{-\varphi_0} \frac{\beta(\alpha - 1) + 3\alpha^2}{\gamma\lambda_{\rm s}^2}. \tag{2.19}$$

The second equality in Eq. (2.19) is only required in the case $\gamma \neq 0$. In fact, in the case $\gamma = 0$ Eqs. (2.17) and (2.18) imply $\alpha = 0$ and $\beta = 2$. In this solution the scale factor and the energy density are both constant while the dilaton evolves in time.

If $\beta = 0$, Eq. (2.15) implies that the dilaton is constant. According to Eq. (2.17):

$$\alpha = \frac{2}{3(\gamma+1)}. \tag{2.20}$$

This is the standard FRW solution obtainable in a general relativistic context when fluid sources are present.

Finally, if $\gamma \neq 0$, the full solution will be

$$\begin{aligned} a(t) &= (-\tau)^{\frac{2\gamma}{1+3\gamma^2}}, \\ \varphi(t) &= \varphi_0 + \frac{6\gamma-2}{1+3\gamma^2}\ln(-\tau), \\ \rho(t) &= \rho_0(-\tau)^{-\frac{6\gamma(\gamma+1)}{1+3\gamma^2}}. \end{aligned} \tag{2.21}$$

The solution (2.21) describes an accelerated expansion ($\dot{a} > 0$ and $\ddot{a} > 0$) *provided* $\gamma < 0$. In this case the solution (2.21) describes the initial, i.e. $t \to -\infty$, asymptotic state typical of the string-driven pre-big bang scenario [11]. Conversely, if $\gamma > 0$ the solution contracts ($\dot{a} < 0$ and $\ddot{a} < 0$) in the string frame metric. This situation is reminiscent of the initial asymptotic state of ekpyrotic scenarios.

If the dilaton is constant the solution, expressed by Eqs. (2.14) and (2.20), contracts for all $-1 < \gamma \leq 1$ and for $t \to -\infty$. Incidentally, notice that a particular subset of these solutions (i.e. where $\gamma = 1$) is the one used in [13] to argue that rotational inhomogeneities are amplified in pre-big bang models.

Of course, the evolution of the solutions described so far can only be applied for $t < 0$, i.e. during the pre-big bang phase. What happens next is determined by the way the singularity problem is addressed which we will do in section 4. For the moment the attention will be concentrated on the evolution of the vector modes during the pre-big bang phase.

Defining, as usual, a conformal time coordinate η (related to the cosmic time as $a(\eta)d\eta = dt$), the fluctuations of the metric (2.7) are

$$\begin{aligned} \delta G_{0i} &= -a^2(\eta)Q_i, \\ \delta G_{ij} &= a^2(\eta)(\partial_i W_j + \partial_j W_i), \end{aligned} \tag{2.22}$$

with $\partial_i Q^i = \partial_i W^i = 0$. For infinitesimal coordinate transformations

$$x^i \to \tilde{x}^i = x^i + \zeta^i, \tag{2.23}$$

preserving the vector nature of the fluctuation (i.e. $\partial_i \zeta^i = 0$), the two variables Q_i and W_i shall change according to the following transformation rules

$$\tilde{Q}_i = Q_i - \zeta_i', \tag{2.24}$$
$$\tilde{W}_i = W_i + \zeta_i, \tag{2.25}$$

where the prime denotes the derivation with respect to the conformal time coordinate η.

From Eqs. (2.24) and (2.25) it can be argued that the quantity $\tilde{W}_i' + \tilde{Q}_i$ is invariant under infinitesimal coordinate transformations (2.23) preserving the vector nature of the fluctuation. Such a quantity is the rotational generalization of the so-called Bardeen potentials [23], which are customarily defined in the case of scalar fluctuations of the geometry.

With these specifications it is useful to perform calculations in a gauge. There are two possible natural gauge choices. A first choice could be to set $\tilde{Q}_i = 0$. In this case the gauge function ζ_i is determined to be

$$\zeta_i(\eta, \vec{x}) = \int^{\eta} Q_i(\eta', \vec{x}) d\eta' + C_i(\vec{x}). \tag{2.26}$$

Since the gauge function is determined up to an arbitrary (space-dependent) constant, this choice of gauge does not completely fix the coordinate system and this occurrence is reminiscent of what happens in the synchronous coordinate system of scalar fluctuations [23]. The gauge choice $\tilde{Q}_i = 0$ was used, for instance, by the authors of Refs. [1] and [2].

Another equally useful choice is the one for which $\tilde{W}_i = 0$. From Eq. (2.25) the gauge function is determined as $\zeta_i = -W_i$ and the gauge freedom, in this case, is completely fixed. Moreover, in this gauge, the gauge-invariant "Bardeen" potential coincides with Q_i and the only non-vanishing entry of the perturbed metric is δG_{0i}. Using the fact that $\delta G^{0i} = -Q^i/a^2$ the Christoffel connections are, to first-order in the amplitude of the rotational fluctuations of the geometry:

$$\begin{aligned}
&\delta\Gamma_{i0}^{0} = \mathcal{H} Q_i, \quad \delta\Gamma_{ij}^{k} = -\mathcal{H} Q^k \delta_{ij}, \quad \delta\Gamma_{00}^{i} = Q^{i\,\prime} + \mathcal{H} Q^i \\
&\delta\Gamma_{ij}^{0} = -\frac{1}{2}(\partial_i Q_j + \partial_j Q_i), \quad \delta\Gamma_{i0}^{j} = \frac{1}{2}(\partial_i Q^j - \partial^j Q_i),
\end{aligned} \tag{2.27}$$

where, as usual, $\mathcal{H} = (\ln a)'$. The first-order form of the Ricci tensors then becomes:

$$\begin{aligned}
&\delta R_{0i} = (\mathcal{H}' + 2\mathcal{H}^2) Q_i - \frac{1}{2}\nabla^2 Q_i, \\
&\delta R_{ij} = -\frac{1}{2}[(\partial_i Q_j + \partial_j Q_i)' + 2\mathcal{H}(\partial_i Q_j + \partial_j Q_i)].
\end{aligned} \tag{2.28}$$

Finally the first-order form of the Einstein tensors will be, in our gauge,

$$\delta \mathcal{G}_{0i} = -(2\mathcal{H}' + \mathcal{H}^2) Q_i - \frac{1}{2}\nabla^2 Q_i, \quad \delta \mathcal{G}_{ij} = \delta R_{ij}. \tag{2.29}$$

The evolution equations of the vector modes of the geometry can be obtained by consistently perturbing, for instance, Eqs. (2.5) and (2.6). The first-order fluctuation of Eq. (2.6) will be

$$\delta R_{\mu\nu} - \delta\Gamma_{\mu\nu}^{\sigma}\partial_{\sigma}\varphi - \frac{1}{2}\frac{\partial W}{\partial\varphi}\delta G_{\mu\nu} = \lambda_{\rm s}^{2}\delta T_{\mu\nu}, \tag{2.30}$$

while the first-order fluctuation of Eq. (2.5) implies

$$\partial_{\mu}\delta T^{\mu\nu} + \delta T^{\nu\alpha}\overline{\Gamma}_{\mu\alpha}^{\mu} + \delta\Gamma_{\mu\alpha}^{\mu}\overline{T}^{\nu\alpha} + \delta T^{\alpha\beta}\Gamma_{\alpha\beta}^{\nu} + \overline{T}^{\alpha\beta}\delta\Gamma_{\alpha\beta}^{\nu} = 0, \tag{2.31}$$

where $\overline{\Gamma}_{\mu\nu}^{\alpha}$ and $\overline{T}^{\alpha\beta}$ denote, respectively, the background values of the connections and of the energy-momentum tensor.

Recall now that $\overline{\Gamma}_{ij}^{0} = \mathcal{H}\delta_{ij}$, $\overline{\Gamma}_{00}^{0} = \mathcal{H}$ and $\overline{\Gamma}_{0i}^{j} = \mathcal{H}\delta_{i}^{j}$; using the results of Eqs. (2.27) and (2.28), the $(0i)$ and (ij) components of Eq. (2.30) lead to

$$\nabla^{2}Q_{i} = -2\lambda_{\rm s}^{2}a^{2}e^{\varphi}(p+\rho)\mathcal{V}_{i}, \tag{2.32}$$

$$Q_{i}' = (\varphi' - 2\mathcal{H})Q_{i}. \tag{2.33}$$

To obtain the precise form of Eq. (2.32), Eq. (2.9) has been used so as to eliminate a term proportional to Q_{i}. In Eq. (2.32), $\mathcal{V}_{i}$ is the divergenceless fluctuation of the velocity field, which arises by perturbing to first-order the energy-momentum tensor of the fluid sources, i.e.

$$\delta T_{0i} = (p+\rho)u_{0}\delta u_{i} - p\delta G_{0i}. \tag{2.34}$$

Recalling now that $u_{0} = a$ (as implied by $G^{\mu\nu}u_{\mu}u_{\nu} = 1$) and defining $\delta u_{i} = a\mathcal{V}_{i}$, Eq. (2.34) leads to

$$\delta T_{0i} = a^{2}(p+\rho)\mathcal{V}_{i} + a^{2}pQ_{i}, \quad \partial_{i}\mathcal{V}^{i} = 0. \tag{2.35}$$

which is the explicit form used to derive Eq. (2.32).

Using the same conventions for the various perturbation variables, the spatial component of Eq. (2.31) implies:

$$\mathcal{V}_{i}' + (1-3\gamma)\mathcal{H}\mathcal{V}_{i} = 0. \tag{2.36}$$

In summary, the Fourier space version of Eqs. (2.32), (2.33) and (2.36) gives

$$k^{2}Q_{i} = 2\lambda_{\rm s}^{2}a^{2}(p+\rho)e^{\varphi}\mathcal{V}_{i}, \tag{2.37}$$

$$Q_{i}' = (\varphi' - 2\mathcal{H})Q_{i}, \tag{2.38}$$

$$\mathcal{V}_{i}' = (3\gamma - 1)\mathcal{H}\mathcal{V}_{i}, \tag{2.39}$$

where, to avoid confusion with the vector indices, the subscript k denoting the Fourier mode of each variable has been suppressed.

From Eq. (2.38) the rotational fluctuation of the geometry becomes

$$Q_i = c_i(k)\frac{e^{\varphi}}{a^2}, \tag{2.40}$$

where $c_i(k)$ is a (k-dependent) integration constant. Inserting Eq. (2.40) into Eq. (2.37) the velocity field becomes

$$\mathcal{V}_i = \frac{k^2 c_i(k)}{2\lambda_{\rm s}^2 a^4 (p+\rho)}. \tag{2.41}$$

It is immediate to see that Eq. (2.41) is consistent with Eq. (2.39).

Thanks to the results obtained so far, it is now possible to discuss the evolution of the rotational fluctuations in the various solutions presented earlier in this section.

Consider first the solution for $\gamma = 0$. As noted above, in this case $a = a_0$ and $\rho = \rho_0$ are both constants but $\varphi = \varphi_0 - 2\ln(-\tau)$. Hence, Eqs. (2.40) and (2.41) imply:

$$\begin{aligned} Q_i &= e^{\varphi_0}\frac{c_i(k) a_0}{\tau^2}, \\ \mathcal{V}_i &= \frac{k^2 a_0^2 c_i(k)}{2\lambda_{\rm s}^2 \rho_0}, \end{aligned} \tag{2.42}$$

showing that while Q_i grows for $\tau \to 0^-$, $\mathcal{V}_i$ stays constant. Furthermore, also $(p+\rho)\mathcal{V}_i$, the velocity contribution to the fluctuation of the energy-momentum tensor, is also constant.

Consider then the case when the dilaton field is constant (or even absent) and the only matter sources are represented by a perfect barotropic fluid. Hence, recalling Eq. (2.20), the evolution of the rotational modes is given by:

$$\begin{aligned} Q_i &\simeq c_i(k)(-\tau)^{-\frac{4}{3(\gamma+1)}}, \\ \mathcal{V}_i &\simeq \frac{k^2 c_i(k)}{2\lambda_{\rm s}^2 \rho_0 (1+\gamma)}(-\tau)^{\frac{2(3\gamma-1)}{3(\gamma+1)}}. \end{aligned} \tag{2.43}$$

Hence, in the case described by Eqs. (2.20) and (2.43) the scale factor contracts and Q_i increases for $-1 < \gamma \leq 1$. Correspondingly, for $\tau \to 0^-$, $\mathcal{V}_i$ *decreases* if $1/3 < \gamma \leq 1$ while it *increases* for $-1 < \gamma < 1/3$. Note that the case $\gamma = 1$ corresponds to the one analysed specifically by [13]. Furthermore, since in this class of solutions the dilaton is constant, the string and Einstein frames coincide. The case $\gamma = 1/3$ is characteristic since $\mathcal{V}_i$ is constant while Q_i still increases.

It should be appreciated that in the case described by Eq. (2.43) the contribution of the velocity field to the perturbed energy-momentum tensor of the sources becomes

$$(p+\rho)\mathcal{V}_i \simeq (-\tau)^{-\frac{8}{3(\gamma+1)}}, \tag{2.44}$$

i.e. $(p+\rho)\mathcal{V}_i$ is sharply increasing for $\gamma > 0$ and for $\tau \to 0^-$. Since $|(p+\rho)\mathcal{V}_i|$ should always be bounded all along the pre-big bang phase, this solution clearly presents another problem.

One possible conclusion would be to infer a general problem of the scenario. On the other hand it is also possible to argue that this type of realization of pre-big bang dynamics is definitely too naive because of the total absence of a dynamical dilaton field.

In fact, if more realistic string-driven solutions are considered, the situation changes. Consider, indeed, the class of solutions described in Eq. (2.21). Inserting Eq. (2.21) into Eqs. (2.40) and (2.41) the result will be

$$Q_i = c_i(k) e^{\varphi_0} (-\tau)^{\frac{2(\gamma-1)}{1+3\gamma^2}}, \tag{2.45}$$

$$\mathcal{V}_i = \frac{k^2 c_i(k)}{2\lambda_{\rm s}^2 \rho_0 (1+\gamma)} (-\tau)^{\frac{2\gamma(3\gamma-1)}{1+3\gamma^2}}. \tag{2.46}$$

As noticed after Eq. (2.21), in this class of solutions, $\gamma < 0$ implies a phase of accelerated expansion for $\tau < 0$. From Eq. (2.45) Q_i increases for all $-1 \leq \gamma \leq 1$. From Eq. (2.46) it can be also concluded that $\mathcal{V}_i$ *decreases* for $\gamma < 0$ and $\gamma > 1/3$ while it *increases* for $0 < \gamma < 1/3$. Again, the case $\gamma = 1/3$ is special since $\mathcal{V}_i$ is constant there.

From Eq. (2.46) the contribution of the rotational fluctuation to the perturbed energy-momentum tensor can be obtained:

$$(p+\rho)\mathcal{V}_i \simeq (-\tau)^{-\frac{2\gamma}{1+3\gamma^2}}, \tag{2.47}$$

which decreases for $\gamma < 0$ while it increases for $\gamma > 0$.

The analysis of the solution (2.21) is interesting since it means that the pre-big bang initial conditions do not lead to a divergence in the perturbed energy-momentum tensor. On the other hand, in the case of contracting initial conditions (i.e. $\gamma > 0$ for $\tau < 0$), $|(p+\rho)\mathcal{V}_i|$ is unbounded and possibly divergent in the limit $\tau \to 0^-$.

Are the results obtained so far conclusive? In some sense yes since it has been shown that, for some specific ranges of the barotropic index, different classes of solutions of the low-energy string effective action seem to indicate that the rotational modes of the geometry and of the fluid sources are led to increase. However, this conclusion cannot be definite unless the whole evolution of the model is carefully specified. The pre-big bang initial conditions must be analytically connected with an appropriate post-big bang solution where, ideally, the dilaton is stabilized and the Universe expands in a decelerated way.

3 Einstein frame description

The analysis of the rotational inhomogeneities of the geometry and of the fluid sources has been conducted, up to now, in the so-called string frame, where the action takes the form

(2.1). In this section we are going to complement the results of section 2 by mentioning the main aspects of the Einstein frame analysis. This task is simplified by the observation that the dilaton only contributes to the evolution of the homogeneous background. The dilaton *fluctuations*, on the contrary, do not affect the evolution equation of the rotational inhomogeneities. This aspect can be appreciated by considering a well-known analogy between a perfect relativistic fluid with stiff equation of state (i.e. $p = \rho$) and a scalar field with negligible potential. This analogy also implies that the peculiar velocity field must be identified with the spatial derivative of the dilaton fluctuations. Notice, however, that when rotational inhomogeneities are included the analogy with the stiff fluid breaks down. While a perfect fluid with stiff equation of state can support a divergenceless velocity field, the "velocity" associated with the scalar degree of freedom is always proportional to $\partial_i \delta\varphi$, $\delta\varphi$ being the dilaton fluctuation. It is then clear that the divergenceless part of the generalized velocity field is always vanishing.

With these necessary specifications, the Einstein frame background solutions can be obtained from the string frame solutions by means of the following local field redefinitions:

$$\tilde{G}_{\mu\nu} = e^{-\varphi} G_{\mu\nu}, \tag{3.1}$$

$$\tilde{T}_{\mu}^{\nu} = e^{2\varphi} T_{\mu}^{\nu}, \tag{3.2}$$

while the dilaton does not change in the transformation between the two frames. The quantities with tilde refer to the Einstein frame while the quantities without tilde refer to the string frame.

From Eqs. (3.1) and (3.2) it easily follows that

$$\tilde{a} = e^{-\varphi/2} a, \tag{3.3}$$

$$\tilde{\rho} = e^{2\varphi} \rho, \tag{3.4}$$

$$\tilde{p} = e^{2\varphi} p. \tag{3.5}$$

Since, as discussed above, the dilaton fluctuations do not contribute to the evolution equations of the vector modes of the geometry, from the evolution equations of the fluctuations in the string frame, i.e. Eqs. (2.37) and (2.38), the Einstein frame equations can be obtained by identifying the perturbation variables through Eqs. (3.1) and (3.2). In formulae we will have

$$\delta\tilde{G}_{\mu\nu} = e^{-\varphi} \delta G_{\mu\nu}, \tag{3.6}$$

$$\delta\tilde{T}_{\mu}^{\nu} = e^{2\varphi} \delta T_{\mu}^{\nu}. \tag{3.7}$$

Recall now that $\delta\tilde{G}_{0i} = -\tilde{a}^2 \tilde{Q}_i$ and $\delta\tilde{T}_i^0 = (\tilde{p} + \tilde{\rho})\tilde{\mathcal{V}}_i$. Hence, it follows from Eqs. (3.6) and (3.7) that

$$\tilde{\mathcal{V}}_i = \mathcal{V}_i, \quad \tilde{Q}_i = Q_i. \tag{3.8}$$

Inserting the result of Eqs. (3.3)–(3.5) and (3.8) into Eqs. (2.37)–(2.39) the evolution equations of the rotational fluctuations can be easily written, in the Einstein frame, as

$$k^2\tilde{Q}_i = (\tilde{p}+\tilde{\rho})\tilde{a}^2\tilde{\mathcal{V}}_i, \tag{3.9}$$

$$\tilde{Q}_i' + 2\tilde{\mathcal{H}}\tilde{Q}_i = 0, \tag{3.10}$$

$$\tilde{\mathcal{V}}_i' = (3\gamma-1)\left(\tilde{\mathcal{H}}+\frac{\varphi'}{2}\right)\tilde{\mathcal{V}}_i, \tag{3.11}$$

where $\tilde{\mathcal{H}} = \tilde{a}'/\tilde{a}$; units of $2\lambda_{\rm s}^2 = 1$ have been used in Eqs. (3.9)–(3.11). An important point concerning the derivation of these equations is that the conformal time coordinate η does not change in the transition from the string to the Einstein frame while the cosmic time coordinate does change as

$$dt_{\rm E} = e^{-\varphi/2}dt_{\rm s}. \tag{3.12}$$

Using Eqs. (3.12) and (3.3) and recalling the definition of the conformal time coordinate in the two frames (i.e. $\tilde{a}(\eta_{\rm E})d\eta_{\rm E} = dt_{\rm E}$ and $a(\eta_{\rm s})d\eta_{\rm s} = dt_{\rm s}$) it can be easily shown, as anticipated, that $\eta_{\rm E} = \eta_{\rm s}$.

The presence of the dilaton field in Eq. (3.11) seems to make the whole system incompatible. This is not the case since, by virtue of the transformation rules (3.2) and (3.4),(3.5), the dilaton will appear explicitly in the covariant conservation equation of the background sources, which is, in the Einstein frame:

$$\tilde{\rho}' + 3\mathcal{H}(\tilde{\rho}+\tilde{p}) + \frac{\varphi'}{2}(3\tilde{\rho}-\tilde{p}) = 0. \tag{3.13}$$

Equation (3.13) can be directly obtained from Eq. (2.11), whose explicit form, in the conformal time parametrization is

$$\rho' + 3\mathcal{H}(\rho+p) = 0. \tag{3.14}$$

Inserting Eqs. (3.4) and (3.5) into Eq. (3.14), Eq. (3.13) is immediately obtained since, from Eq. (3.3), $\mathcal{H} = \tilde{\mathcal{H}} + \varphi'/2$.

Thus, from Eq. (3.9) and (3.10):

$$\tilde{Q}_i = \frac{\tilde{c}_i(k)}{\tilde{a}^2}, \tag{3.15}$$

$$\tilde{\mathcal{V}}_i = \frac{k^2\tilde{c}_i(k)}{\tilde{a}^4(\tilde{\rho}+\tilde{p})} \equiv \frac{k^2\tilde{c}_i(k)}{(\gamma+1)}\tilde{a}^{3\gamma-1}e^{\varphi(3\gamma-1)/2}. \tag{3.16}$$

The second equality in Eq. (3.16) follows by replacing $\tilde{\rho}$ with the result of the direct integration of Eq. (3.13). Equations (3.15) and (3.16) are the ones obtained in [13] in the rather specific case of constant dilaton field.

Finally the Einstein-frame version of the solutions with generic barotropic index reported in Eqs. (2.21) are

$$\tilde{a} \simeq \left(-\frac{\eta}{\eta_0}\right)^{-\frac{\gamma-1}{3\gamma^2-2\gamma+1}},$$
$$\varphi = \tilde{\varphi} \simeq \varphi_0 + \frac{2(3\gamma-1)}{3\gamma^2-2\gamma+1}\ln\left(-\frac{\eta}{\eta_0}\right),$$
$$\tilde{\rho} \simeq \left(-\frac{\eta}{\eta_0}\right)^{\frac{6\gamma(1-\gamma)-4}{3\gamma^2-2\gamma+1}}, \qquad (3.17)$$

where the normalization constants have been dropped and only the relevant time dependence reported.

Let us now compare the evolution of the vector modes of the geometry in the case of the solution given in Eq. (2.21) which is written in the cosmic time coordinate, i.e. $\tau = t/t_0$. This solution can be translated in the conformal time parametrization by recalling that $a(\eta)d\eta = dt$. Applying the mentioned differential relation we find

$$(-\tau) \simeq (-\eta)^{\frac{3\gamma^2+1}{1+3\gamma^2-2\gamma}}. \qquad (3.18)$$

From Eqs. (2.45) and (2.46) we can then derive the evolution of the vector modes in the *string frame* and in the conformal time coordinate, i.e.

$$\mathcal{Q}_i \simeq (-\eta)^{\frac{2(\gamma-1)}{3\gamma^2-2\gamma+1}},$$
$$\mathcal{V}_i \simeq (-\eta)^{\frac{2\gamma(3\gamma-1)}{3\gamma^2-2\gamma+1}}. \qquad (3.19)$$

Equation (3.19), we repeat, gives the solution in the string frame.

Now, let us compute *the same evolution* in the Einstein frame. Equations (3.17) are the Einstein frame version of Eqs. (2.21). The evolution of the vector modes in the Einstein frame is then obtained by inserting Eqs. (3.19) into Eqs. (3.15) and (3.16). Comparing the obtained expressions with Eq. (3.19), we do obtain that

$$\tilde{\mathcal{V}}_i = \mathcal{V}_i, \qquad \tilde{Q}_i = Q_i. \qquad (3.20)$$

This is exactly the result anticipated, on a general ground, in Eq. (3.8). This means that, even if in the two frames the evolution of the scale factors is different, the evolution of the fluctuations will be the same in the two frames. The rationale for this result is that the transformation of the background solution is compensated by the transformation of the evolution equations leading, as expected, to the complete equivalence of the two approaches. Notice also, as a final remark, that the complete equivalence is also a consequence of the fact that, unlike the cosmic time coordinate, the conformal time coordinate *is invariant* when passing from String to Einstein frame (and viceversa). This property has been swiftly re-derived below Eq. (3.12) and is well known.

4 Decaying rotational inhomogeneities

In order to assess the phenomenological relevance of the present exercise as well as of related ideas [13], it is important to discuss what the fate of the rotational inhomogeneities is. It should be understood to what extent the rotational modes of the geometry will persist during the post-big bang evolution. A closely related issue would be to check if the rotational modes of the geometry and of the sources may diverge around the time of the transition between pre- and post-big bang.

In the presence of a dynamical dilaton field, it is possible to find completely regular solutions describing the transition from the pre- to the post-big bang evolution. These solutions may well include fluid sources and can be obtained when the dilaton potential is a function of $e^{-\overline{\varphi}}$, defined as

$$e^{-\overline{\varphi}(x)} = \frac{1}{\lambda_{\rm s}^3}\int d^4y\sqrt{-G(y)}e^{-\varphi(y)}\sqrt{(\partial\varphi)^2_y}\delta(\varphi(x)-\varphi(y)). \tag{4.1}$$

These solutions were explored through various steps [24, 25] (see also [11]).

In [26] it has been shown that a generally covariant action can be written for these models. The evolution of the scalar and tensor modes of the geometry has been discussed in detail and it has been concluded that the evolution of these inhomogeneities is completely regular all along the different stages of the model [27]. In [28] it was demonstrated that back-reaction effects of massless particles can naturally induce a hot phase dominated by radiation with a stabilized dilaton field (see also [29] for some earlier attempts to stabilize the dilaton in a radiation-dominated post-big bang evolution).

In this section we are going to discuss the evolution of the rotational modes in the framework of the same regularization scheme as introduced in [26, 27]. Consider indeed the action [26]

$$S = -\frac{1}{2\lambda_{\rm s}^2}d^4x\sqrt{-G}e^{-\varphi}[R+(\partial\varphi)^2+V]+S_{\rm m}, \tag{4.2}$$

where $V \equiv V(e^{-\overline{\varphi}})$.

Following the results of Refs. [6, 26] the relevant equations of motion can be written as

$$\mathcal{G}_{\mu\nu} + \nabla_\mu\nabla_\nu\varphi + \frac{1}{2}G_{\mu\nu}[(\partial\varphi)^2 - 2\nabla^2\varphi - V] - \frac{1}{2}e^{-\varphi}\sqrt{(\partial\varphi)^2}\gamma_{\mu\nu}I_1 = e^{\varphi}\lambda_{\rm s}^2 T_{\mu\nu}, \tag{4.3}$$

$$\nabla_\mu T^{\mu\nu} = 0, \tag{4.4}$$

where

$$I_1 = \frac{1}{\lambda_{\rm s}^3}\int d^4y(\sqrt{-G}V')_y\delta(\varphi(x)-\varphi(y)), \tag{4.5}$$

and

$$\gamma_{\mu\nu} = G_{\mu\nu} - \frac{\partial_\mu\varphi\partial_\nu\varphi}{(\partial\varphi)^2}. \tag{4.6}$$

In Eq. (4.5) $V' = \partial V/\partial(e^{-\overline{\varphi}})$. In the homogeneous limit the evolution equations of the geometry are found to be [26, 27]

$$\dot{\overline{\varphi}}^2 - 3H^2 - V = 2\lambda_{\rm s}^2 e^{\overline{\varphi}}\overline{\rho} \tag{4.7}$$

$$\dot{H} = H\dot{\overline{\varphi}} + \lambda_{\rm s}^2 e^{\overline{\varphi}}\overline{p}, \tag{4.8}$$

$$2\ddot{\overline{\varphi}} - \dot{\overline{\varphi}}^2 - 3H^2 + V - \frac{\partial V}{\partial\overline{\varphi}} = 0, \tag{4.9}$$

$$\dot{\overline{\rho}} + 3H\overline{p} = 0. \tag{4.10}$$

Notice that, in the homogeneous limit, from Eq. (4.21), $e^{-\overline{\varphi}} = e^{-\varphi}a^3$, where we have absorbed into φ the dimensionless constant $-\ln\left(\int d^3y/\lambda_{\rm s}^3\right)$ associated with the (finite) comoving spatial volume. In Eqs. (4.7)–(4.10) the reduced energy and pressure densities

$$\overline{\rho} = a^3\rho, \qquad\qquad \overline{p} = a^3 p \tag{4.11}$$

have been defined.

The evolution equations of the vector modes of the geometry can be obtained, following the same notation as in section 2, from

$$\delta\mathcal{G}_{\mu\nu} - \delta\Gamma_{\mu\nu}^{\sigma}\partial_\sigma\varphi + \frac{1}{2}\delta G_{\mu\nu}[(\partial\varphi)^2 - 2\nabla^2\varphi - V] - \frac{1}{2}e^{-\varphi}\sqrt{(\partial\varphi)^2}\delta\gamma_{\mu\nu}I_1 = \lambda_{\rm s}^2\delta T_{\mu\nu}. \tag{4.12}$$

The perturbed form of the covariant conservation equation is the same as the one reported in Eq. (2.31). The fluctuations of I_1 do not contribute to the vector fluctuations. Moreover, in the gauge $\tilde{W}_i = 0$, $\delta\gamma_{0i} = -a^2 Q_i$. Using now Eqs. (2.27) and (2.29), the evolution equations of the vector modes can be written, in Fourier space, as:

$$Q_i' = (\overline{\varphi}' + \mathcal{H})Q_i, \tag{4.13}$$

$$k^2\mathcal{V}_i = 2\lambda_{\rm s}^2(\overline{\rho} + \overline{p})a^2 e^{\overline{\varphi}}, \tag{4.14}$$

$$\mathcal{V}_i' = (3\gamma - 1)\mathcal{H}\mathcal{V}_i. \tag{4.15}$$

Equations (4.8) and (4.9) can be written, in the case $V = -V_0 e^{2\overline{\varphi}}$ as

$$\frac{d}{dt}(He^{-\overline{\varphi}}) = \frac{\gamma}{2}\overline{\rho}, \tag{4.16}$$

$$\frac{d}{dt}(e^{-\overline{\varphi}}\dot{\overline{\varphi}}) = -\frac{\overline{\rho}}{2}. \tag{4.17}$$

In order to obtain Eq. (4.17), Eq. (4.7) has been used to eliminate $3H^2$ from Eq. (4.9). Notice also that, in Eqs. (4.16) and (4.17), the system of $2\lambda_{\rm s}^2 = 1$ units has been adopted.

Let us now define the new time coordinate as

$$\frac{dx}{dt} = \overline{\rho}\frac{L}{2}, \tag{4.18}$$

where L is a constant (dimensionful) parameter. Equations (4.16) and (4.17) can be integrated once giving

$$H = \frac{\gamma}{L} x e^{\overline{\varphi}}, \tag{4.19}$$

$$\dot{\overline{\varphi}} = -\frac{x}{L} e^{\overline{\varphi}}, \tag{4.20}$$

having set to zero the integration constants.

Inserting Eqs. (4.19) and (4.20) into Eq. (4.7) we will obtain

$$e^{\overline{\varphi}} = \frac{\overline{\rho} L^2}{(1-3\gamma^2)x^2 + V_0 L^2}. \tag{4.21}$$

Clearly $e^{\overline{\varphi}}$ is positive-definite and non-singular iff

$$c^2 = V_0 L^2 > 0, \qquad b^2 = 1 - 3\gamma^2 > 0. \tag{4.22}$$

This means that the regular solutions obtained in this way will be defined for a reasonable set of barotropic indices, namely $|\gamma| < 1/\sqrt{3}$. Inserting Eq. (4.21) into Eqs. (4.16) and (4.17), and using the new time coordinate x as defined in Eq. (4.18), the full solution can be written as

$$a(x) = (b^2x^2 + c^2)^{\frac{\gamma}{b^2}}, \tag{4.23}$$

$$\overline{\varphi}(x) = \overline{\varphi}_0 - \frac{1}{b^2}\ln(b^2x^2 + c^2), \tag{4.24}$$

$$\overline{\rho} = \overline{\rho}_0 (b^2x^2 + c^2)^{-3\frac{\gamma^2}{b^2}}. \tag{4.25}$$

Now that the explicit solution has been obtained in terms of x, the relation between x and t can be found from Eq. (4.18) by direct insertion of Eq. (4.25), with the result

$$\frac{dx}{dt} = \frac{L\overline{\rho}_0}{2} \frac{1}{(b^2x^2 + c^2)^{3\frac{\gamma^2}{b^2}}}. \tag{4.26}$$

From Eq. (4.26) it also follows that

$$|bx| \simeq |bt|^{\frac{b^2}{b^2 + 6\gamma^2}}, \tag{4.27}$$

for $|x| \to \infty$ and $t \to \infty$.

From Eqs. (2.37) and (2.38) the evolution of the vector modes easily follows, using Eqs. (4.23)–(4.25):

$$Q_i = c_i(k) e^{\overline{\varphi}_0} (b^2x^2 + c^2)^{\frac{\gamma - 1}{b^2}}, \tag{4.28}$$

$$\mathcal{V}_i = \frac{k^2 c_i(k)}{(1+\gamma)\overline{\rho}_0} (b^2x^2 + c^2)^{\frac{\gamma(3\gamma - 1)}{b^2}}. \tag{4.29}$$

In order to understand the evolution of the vector modes it should be clear that since $b^2 > 0$, then $|\gamma| < 1/\sqrt{3}$. In this case the exponent appearing at the right-hand side of Eq. (4.28) is always negative, so that Q_i increases for $x < 0$ and decreases for $x > 0$, being regular for $x = 0$. From Eq. (4.29) it can be argued that the velocity field is decreasing for $x < 0$ if (and only if) $-1/\sqrt{3} < \gamma < 0$ and $1/3 < \gamma < 1/\sqrt{3}$. On the contrary, if $0 < \gamma < 1/3$, $\mathcal{V}_i$ increases for $x < 0$.

For $x > 0$, $\mathcal{V}_i$ increases for $-1/\sqrt{3} < \gamma < 0$ and $1/3 < \gamma < 1/\sqrt{3}$, while it decreases for $0 < \gamma < 1/3$. The case $\gamma = 1/3$ leads to constant $\mathcal{V}_i$.

From the example discussed so far, it seems that for $-1/\sqrt{3} < \gamma < 0$ the rotational modes of the sources decrease for $x < 0$ but increase for $x > 0$. This effect is a direct consequence of the evolution of $a(x)$ as reported in Eq. (4.23). Notice, indeed, that from Eq. (4.23) and (4.27) $a(t)$ describes an accelerated expansion for $t < 0$ and $\gamma < 0$, but it is contracting for $t > 0$ and the same choice of γ. Analogously, if $\gamma > 0$, $a(t)$ will contract during the pre-big bang phase and it will expand in the post-big bang phase.

It does not seem realistic to have a contracting evolution in the post-big bang phase, which should represent, in some general sense, the early phases of our present Universe below the Planck/string energy scale.

A more realistic example can be formulated in the case of $p = 0$ and for the following potential

$$V = -V_0 e^{\overline{\varphi}} - V_1 e^{4\overline{\varphi}}. \tag{4.30}$$

Since $p = 0$ (and also, clearly, $\overline{p} = 0$), Eq. (4.10) implies $\overline{\rho} = \overline{\rho}_0$, i.e. that $\overline{\rho}$ is constant. Thus, Eqs. (4.7)–(4.9) can be integrated to give

$$\begin{aligned} H &= \frac{1}{\sqrt{3}} \frac{1}{\sqrt{t^2 + t_0^2}}, \\ \dot{\overline{\varphi}} &= \frac{t}{t^2 + t_0^2}, \end{aligned} \tag{4.31}$$

with $V_0 = \overline{\rho}_0$ and $t_0^2 = e^{4\overline{\varphi}_0} V_1$. In the case given by Eq. (4.31) the evolution of the rotational inhomogeneities is given by

$$\begin{aligned} \mathcal{V}_i &= \frac{k^2 c_i(k)}{\overline{\rho}_0} (t + \sqrt{t^2 + t_0^2})^{-1/\sqrt{3}}, \\ Q_i &= e^{\overline{\varphi}_0} \frac{(t + \sqrt{t^2 + t_0^2})^{1/\sqrt{3}}}{\sqrt{t^2 + t_0^2}}. \end{aligned} \tag{4.32}$$

In this case, again, the rotational modes are regular. The variable Q_i increases for $t < 0$ and decreases for $t > 0$, being regular for $t = 0$. The variable $\mathcal{V}_i$ is always decreasing.

An even more realistic example will now be presented. In this case the evolution will interpolate between a pre-big bang phase for $t \to -\infty$ and a radiation-dominated phase for $t \to +\infty$. Consider, indeed, the case in which γ is a function of x, interpolating between $-1/3$ (for $x \to -\infty$) and $1/3$ (for $x \to +\infty$). Consider, in particular, the case

$$\gamma(x) = \frac{1}{3}\frac{x}{\sqrt{x^2 + x_1^2}}. \tag{4.33}$$

Equations (4.16) and (4.17) can also be integrated, in the case of the $\gamma(x)$ given in Eq. (4.33), with a potential $V = -V_0 e^{2\overline{\varphi}}$; the solution can be written as

$$\begin{aligned} a(x) &= x + \sqrt{x^2 + x_1^2}, \\ \overline{\varphi} &= \overline{\varphi}_0 - \frac{3}{2}\ln(x^2 + x_1^2), \\ \overline{\rho} &= \frac{\overline{\rho}_0}{\sqrt{x^2 + x_1^2}}, \end{aligned} \tag{4.34}$$

with

$$V_0 L^2 = x_1^2, \quad \overline{\rho}_0 L^2 = \frac{2}{3} e^{\overline{\varphi}_0}. \tag{4.35}$$

From Eq. (4.34) the evolution of the vector modes can be found to be

$$\mathcal{V}_i = \frac{3k^2 c_i(k)}{\overline{\rho}_0}\frac{x^2 + x_1^2}{(x + \sqrt{x^2 + x_1^2})(x + 3\sqrt{x^2 + x_1^2})}, \tag{4.36}$$

$$Q_i = c_i(k) e^{\overline{\varphi}_0}\frac{x + \sqrt{x^2 + x_1^2}}{(x^2 + x_1^2)^{3/2}} \tag{4.37}$$

In order to have a clear physical interpretation of the solution, we can notice that, in the asymptotic regions, i.e. $|t| \to \infty$,

$$\begin{aligned} -x &\simeq \sqrt{-t}, \quad \text{for } t \to -\infty, \\ x &\simeq \sqrt{t}, \quad \text{for } t \to \infty. \end{aligned} \tag{4.38}$$

Then looking at the specific form of $a(x)$ as reported in Eq. (4.34) it can be appreciated that $a(t) \simeq (-t)^{-1/2}$ (for $t \to -\infty$) and $a(t) \simeq t^{1/2}$ (for $t \to +\infty$).

According to Eq. (4.36), the velocity field always decreases during the pre-big bang epoch and also later on. Notice in fact that

$$\begin{aligned} \mathcal{V}_i &\simeq \frac{c_i k^2}{\overline{\rho}_0} x^2, \quad \text{for } x \to -\infty, \\ \mathcal{V}_i &\simeq \frac{c_i k^2}{\overline{\rho}_0}, \quad \text{for } x \to \infty. \end{aligned} \tag{4.39}$$

If we set initial pre-big bang initial conditions for $x \to -\infty$, $\mathcal{V}_i$ will then decrease and then set to a constant value. From Eq. (4.37) the variable Q_i increases for $x < 0$ but it decreases, as anticipated, for $x > 0$.

5 Concluding remarks

In this paper the evolution of the rotational inhomogeneities of the geometry and of the fluid sources has been analysed in the framework of four-dimensional pre-big bang models, in both the string and Einstein frames. The main results can be summarized as follow

- in the case of minimal (dilaton-driven models) the rotational inhomogeneities are totally absent;
- if fluid sources are added during the pre-big bang phase, the solution of the system shows that the rotational inhomogeneities can grow for negative values of the barotropic index, while for positive values rotational inhomogeneities decay ;
- the fate of the vector modes has been analysed in various non-singular models, where the evolution of the scalar and tensor modes of the geometry is well defined and non-singular;
- if the evolution of the geometry and of the dilaton is regular, then, as expected, the vector modes are never divergent;
- for realistic examples, where the Universe inflates during the pre-big bang epoch and turns into radiation later on, the rotational inhomogeneities increase in the pre-big bang, but swiftly decay later on.
- in this systematic study, all the known string cosmological solutions with fluid sources in four space-time dimensions have been scrutinized.

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